On the notion of conditional symmetry of differential equations

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Abstract

Symmetry properties of PDE's are considered within a systematic and unifying scheme: particular attention is devoted to the notion of conditional symmetry, leading to the distinction and a precise characterization of the notions of "true" and "weak" conditional symmetry. Their relationship with exact and partial symmetries is also discussed. An extensive use of "symmetry-adapted" variables is made; several clarifying examples, including the case of Boussinesq equation, are also provided.

1 Introduction

In the study of general aspects of differential equations, and also in the concrete problem of finding their explicit solutions, a fundamental role is played, as well known, by the analysis of symmetry properties of the equations. In addition to the classical notion of Lie "exact" symmetries (see e.g. [1]–[7]), an important class of symmetries is given by the "conditional symmetries" (or "nonclassical symmetries"), introduced and developed by Bluman and Cole [8, 9], Levi and Winternitz [10, 11], Fushchych [12, 13] and many others (see e.g. [5, 11]).

In this paper we will be concerned with partial differential equations (PDE) and with the above mentioned types of symmetries, and also with the notion of "partial symmetry", as defined in [14]: in the context of a simple comprehensive scheme, we will distinguish different notions of conditional symmetry, with a precise characterization of their properties and a clear comparison with other types of symmetries.

An extensive use will be made of the "symmetry-adapted" variables (also called "canonical coordinates", see e.g. [2, 7] and also [15]), which reveal to be extremely useful; several clarifying examples will be also provided, including the case of Boussinesq equation, which offers good examples for all the different notions of symmetry considered in this paper (see also [16]).

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For the sake of simplicity, only "geometrical" or Lie point-symmetries will be considered, although the relevant results could be extended to more general classes of symmetries, as generalized or Bäcklund, potential or nonlocal symmetries, whose importance is well known and also recently further emphasized (see e.g. [17]–[20]).

2 Preliminary statements

Let us start with a preliminary Lemma, simple but important for our applications. In view of this, the notations are chosen similar, as far as possible, to those used below.

Lemma 1 *Consider a system of n equations for the n functions $y_a = y_a(s)$ ($a = 1, \dots, n$; $s \in \mathbf{R}$) of the form (sum over repeated indices)*

$$\frac{\mathrm{d}y_a}{\mathrm{d}s} = G_{ab}(s, y)\, y_b \tag{1}$$

where G_{ab} are $n \times n$ given functions of s and of $y \equiv (y_1, \dots, y_n)$, which are assumed regular enough (e.g., analytic in a neighbourhood of $s = 0$, $y = 0$). Then, any solution of (1) can be written, in a neighbourhood of $s = 0$,

$$y_a(s) = R_{ab}(s)\, \kappa_b \tag{2}$$

where $\kappa \equiv (\kappa_1, \dots, \kappa_n)$ are constants, and R_{ab} are regular functions with

$$R_{ab}(0) = \delta_{ab}$$

(then, $\kappa_a = y_a(0)$). Reciprocally, for any solution $y_a(s)$, there are regular functions $S_{ab}(s)$ such that

$$\kappa_a = S_{ab}(s)\, y_b(s)\ . \tag{3}$$

Proof. The result is nearly trivial if G_{ab} do not depend on y. In the general case, let $y_a = \overline{y}_a(s)$ denote any given solution of (1) in a neighbourhood of $s = 0$, determined by n initial values $\kappa_a = \overline{y}_a(0)$ (we omit to write explicitly the dependence on the κ). Let us put

$$K_a(s) = S_{ab}(s)\, \overline{y}_b(s)$$

where S_{ab} are functions to be determined; we then get (with $' = \mathrm{d}/\mathrm{d}s$)

$$K'_a = S'_{ab}\, \overline{y}_b + S_{ab}\, \overline{y}'_b = \Big(S'_{ab} + S_{ac}\, G_{cb}(s, \overline{y}(s))\Big)\, \overline{y}_b \tag{4}$$

Consider now the equation for the matrix S, with clear notations in matrix form,

$$S' = -SG \tag{5}$$

where it is understood that the generic solution $\overline{y}$ is replaced in G by its expression depending on s (and on κ, of course). Now, eq. (5) always admits a solution

S – as well known (see e.g. [21]) – which can be characterized as a fundamental matrix for the associated "adjoint" system $\zeta'_a = -\zeta_b G_{ba}$. In particular, this fundamental matrix can be constructed assuming as initial value at $s = 0$ the matrix $S(0) = I$. Therefore, choosing this matrix S, one gets from (4) $K_a = \text{const} = K_a(0) = S_{ab}(0)\overline{y}_b(0) = \overline{y}_a(0) = \kappa_a$, and $\kappa_a = S_{ab}(s)\,\overline{y}_b(s)$. The matrix $S(s)$ can be locally inverted, giving for any solution $y_a(s)$, $y_a = S^{-1}_{ab}\,\kappa_b := R_{ab}\,\kappa_b$ with $R = R(s)$ and $R(0) = I$.

Remark 1. As is clear from the proof, S and R also depend on the initial values κ which indeed determine the generic solution $\overline{y}(s)$; the only relevant points here are the "factorization" of the κ as in (2) and the form (3), i.e. the possibility of obtaining $s-$independent "combinations" (with coefficients S depending on s) of the components of each solution. In our applications, the functions G_{ab} will also depend on some other parameters; then all results hold true, but clearly S, R and κ turn out to be functions of these additional quantities.

In the following, we will consider systems of PDE's, denoted by

$$\begin{array}{c} \Delta \equiv \Delta_a(x, u^{(m)}) \,=\, 0 \quad , \qquad\quad a =, \ldots, \nu\,, \\ u \equiv (u_1, \ldots, u_q) \quad ; \qquad x \equiv (x_1, \ldots, x_p) \end{array} \tag{6}$$

for the q functions $u_\alpha = u_\alpha(x)$ of the p variables x_i, where $u^{(m)}$ denotes the functions u_α together with their x derivatives up to the order m, with usual notations and assumptions (as stated, e.g., in [2]). In particular, we will always assume that all standard smoothness properties and the maximal rank condition are satisfied. As anticipated, only Lie point-symmetries will be considered, with infinitesimal generator given by vector field

$$X \,=\, \xi_i(x,u)\frac{\partial}{\partial x_i} + \varphi_\alpha(x,u)\frac{\partial}{\partial u_\alpha} \tag{7}$$

To simplify notations, we shall denote by X^* the "appropriate" prolongation of X for the equation at hand, or – alternatively – its infinite prolongation (indeed, only a finite number of terms will appear in calculations).

For completeness, and – even more – for comparison with the subsequent Definition 2, let us start with the following (completely standard) definition (cf. [2]).

Definition 1 *A (nondegenerate) system of PDE $\Delta_a(x, u^{(m)}) \,=\, 0$ is said to admit the Lie point-symmetry generated by the vector field X (or to be symmetric under X) if the following condition*

$$X^*(\Delta)|_{\Delta=0} \,=\, 0 \tag{8}$$

is satisfied, or – equivalently (at least under mild hypothesis, see [2]) – if there are functions $G_{ab}(x, u^{(m)})$ such that

$$(X^*(\Delta))_a \,=\, G_{ab}\,\Delta_b\;. \tag{9}$$

Let us also give this other definition.

Definition 2 *A system of PDE as before is said to be* invariant *under a vector field X if*

$$X^*(\Delta) \,=\, 0 \,. \tag{10}$$

For instance, the Laplace equation $u_{xx} + u_{yy} = 0$ is *invariant* under the rotation symmetry generated by $X = y\partial/\partial x - x\partial/\partial y$; the heat equation $u_t = u_{xx}$ is *symmetric* but *not* invariant under

$$X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} \tag{11}$$

indeed one has $X^*(u_t - u_{xx}) = -x(u_t - u_{xx})$.

3 Symmetric and Invariant Equations

Let us introduce a first simplification: we will assume that the vector fields X are "projectable", or – more explicitly – that the functions ξ in (7) do not depend on u, as often happens in the study of PDE's. This strongly simplifies calculations, especially in the introduction of the more "convenient" or "symmetry-adapted" variables, and allows a more direct relationship between symmetries and symmetry-invariant solutions, as discussed in [22].

A first result, concerning "exact" (to be distinguished from conditional or partial, see below) symmetries is the following (see also [2, 7]).

Theorem 1 *Let $\Delta = 0$ be a nondegenerate system of PDE, symmetric under a vector field X, according to Def. 1. Then, there are new p+q variables s, z and v, with $s \in \mathbf{R}$, $z \in \mathbf{R}^{p-1}$, $v \equiv (v_1(s,z), \ldots, v_q(s,z))$, and a new system of PDE's, say $K = 0$, with $K_a = S_{ab}(s,z,v^{(m)})\,\widetilde{\Delta}_b(s,z,v^{(m)})$ [where $v^{(m)}$ stands for $v(s,z)$ and its derivatives with respect to s and z, and $\widetilde{\Delta} = \widetilde{\Delta}(s,z,v^{(m)})$ is Δ when expressed in terms of the new variables s, z, v], which is locally* equivalent *to the initial system and is* invariant *(as in Def. 2) under the symmetry $X = \partial/\partial s$, i.e. $K_a = K_a(z, v^{(m)})$.*

Proof. Given X, one introduces "canonical variables" s, z, defined by

$$X\ s \,\equiv\, \xi_i \frac{\partial s}{\partial x_i} + \varphi_\alpha \frac{\partial s}{\partial u_\alpha} \,=\, 1 \quad ; \quad X\ z_k \,=\, 0 \quad (k = 1, \ldots, p-1) \tag{12}$$

One first considers the subset of characteristic equations $\mathrm{d}x_i/\xi_i = \mathrm{d}s$ which do not contain the variables u_α, and finds the variable s together with the $X-$invariant variables z_k. Then, using the characteristic equations $\mathrm{d}x_i/\xi_i = \mathrm{d}u_\alpha/\varphi_\alpha$, one finds the q invariant quantities v_α, and expresses the u_α in terms of v_α and of the new independent variables s, z_k. Once written in these coordinates, the symmetry field and all its prolongations are simply given by

$$X = X^* = \frac{\partial}{\partial s} \tag{13}$$

whereas the symmetry condition becomes $\partial\widetilde{\Delta}/\partial s|_{\widetilde{\Delta}=0} = 0$ or

$$\frac{\partial}{\partial s}\widetilde{\Delta}_a = G_{ab}\widetilde{\Delta}_b \tag{14}$$

An application of Lemma 2.1, where the role of y is played here by $\widetilde{\Delta}$ and that of κ by K, shows that there are suitable "combinations" $K_a = S_{ab}\widetilde{\Delta}_b$ of the $\widetilde{\Delta}_a$ which do not depend *explicitly* on s, i.e. $(\partial/\partial s)K = 0$.

This result can be compared with an analogous result presented in [23], where however the point of view is different (i.e., constructing equations with a prescribed algebra of symmetries).

Example 1. Consider the quite trivial system of PDE for $u = u(x,y)$

$$u_{xx} + u_{yy} + u_{xxx} = 0 \tag{15a}$$

$$u_{xxx} = u_{xxy} = u_{xyy} = u_{yyy} = 0 \tag{15b}$$

This system admits the rotation symmetry $X = y\partial/\partial x - x\partial/\partial y$, although none of the equations above is invariant or symmetric under rotations. The variables s,z are in this case obviously the polar variables θ, r, and $X = X^* = \partial/\partial\theta$; it is now easy to construct combinations of the above equations for $v = v(r,\theta)$ which are invariant under $\partial/\partial\theta$: e.g.

$$yu_{xxx} - xu_{xxy} + yu_{xyy} - xu_{yyy} = -r^{-2}(rv_{r\theta} + r^2 v_{rr\theta} + v_{\theta\theta\theta}) = 0$$

$$xu_{xxx} + yu_{xxy} + xu_{xyy} + yu_{yyy} = (r^{-2})(-rv_r + r^2 v_{rr} - 2v_{\theta\theta} + r^3 v_{rrr} + rv_{r\theta\theta}) = 0$$

It can be remarked that considering equation $(15a)$, together with only the first one of the $(15b)$, i.e. $u_{xxx} = 0$, one obtains a system which is *not* symmetric under rotations, although the equation $u_{xxx} = 0$ expresses the vanishing of the "symmetry breaking term" in $(15a)$. As a consequence, the system of these two equations would admit solutions, e.g. $u = x^2 y - y^3/3$, which are *not* transformed by rotations into other solutions. $\odot$

Example 2. In the example of heat equation mentioned at the end of previous Section, choosing the variables $s = x/2t$, $z = t$ and with $u = \exp(-zs^2)v(s,z)$ as determined by the symmetry vector field (11), the equation is transformed into the *equivalent* equation for $v = v(s,z)$

$$4z^2 v_z + 2zv - v_{ss} = 0$$

($v_s = \partial v/\partial s$, etc.) which indeed does not depend explicitly on s and therefore is *invariant* under the symmetry $X = \partial/\partial s$ (but does contain a function v depending on s). Now looking for solutions with $v_s = 0$, i.e. with $v = w(z)$, one obtains the known reduced equation $2zw_z + w = 0$ (see [2]). $\odot$

It should be emphasized that the result in Theorem 1 is not the same as (but is related to, and includes in particular) the well known result concerning the reduction of the given PDE to $X-$invariant equations for the variables $w(z)$:

indeed, introducing the new "symmetry-adapted" variables s, z and $v(s,z)$, we have transformed the equation into a locally *equivalent* equation for $v(s,z)$. If one now *further* assumes that $\partial v/\partial s = 0$, i.e. if one looks for the $X-$invariant solutions $v = w(z)$, then the equations $K_a = 0$ become a system of equations

$$K_a^{(0)}(z, w^{(m)}) = 0 \tag{16}$$

involving only the variables z and functions depending only on z (see [24] for a detailed discussion on the reduction procedure). In particular, in the case of a single PDE for a single unknown function depending on two variables, the PDE is reduced to an ODE, as well known, and as in Example 2 above.

4 Conditional Symmetries, in "true" and "weak" sense

The above approach includes in a completely natural way some other important situations. It is known indeed that, by means of the introduction of the notion of conditional symmetry (CS), one may obtain other solutions which turn out to be invariant under these "nonclassical" symmetries [5], [8]–[13]. But there are *different* types of CS, and it is useful to distinguish these different notions and to see how they can be fitted in this scheme.

To avoid unessential complications with notations, we will consider from now on only the case of a single PDE $\Delta = 0$ for a single unknown function $u(x)$; the extension to the general cases is in principle straightforward.

As well known, a vector field X is said to be a conditional symmetry for the equation $\Delta = 0$ if X is an "exact" Lie point-symmetry for the system

$$\Delta \, = \, 0 \qquad ; \qquad X_Q u \equiv \varphi - \xi_i \frac{\partial u}{\partial x_i} \, = \, 0 \tag{17}$$

where X_Q is the symmetry written in "evolutionary form" [2]. The second equation in (17), indicating that we are looking for solutions *invariant* under X, is automatically symmetric under X; we have then only to impose

$$X^*(\Delta)\Big|_\Sigma \, = \, 0 \tag{18}$$

where Σ is the set of the simultaneous solutions of the two equations (17), plus (possibly) some differential consequences of the second one (see [2], [25]–[31] for a precise and detailed discussion on this point and the related notion of degenerate systems of PDE). In the canonical variables s, z and $v = v(s,z)$ determined by the vector field X, the invariance condition $X_Q v = 0$ becomes

$$\frac{\partial v}{\partial s} \, = \, 0 \tag{19}$$

and the condition of CS (18) takes the simple form (let us now retain for simplicity the same notation Δ, instead of $\widetilde{\Delta}$, also in the new coordinates)

$$\frac{\partial \Delta}{\partial s}\Big|_\Sigma \, = \, 0 \tag{20}$$

here Σ stands for the set of the simultaneous solutions of $\Delta = 0$ and $v_s = 0$, together with the derivatives of v_s with respect to all variables s and z_k. Using the global notation $v_s^{(\ell)}$ to indicate v_s, v_{ss}, v_{sz_k} etc., the CS condition (20) becomes then equivalently, according to Def. 1, and with clear notations,

$$\frac{\partial}{\partial s}\Delta \;=\; G(s,z,v^{(m)})\,\Delta + \sum_\ell H_\ell(s,z,v^{(m)})\,v_s^{(\ell)} \tag{21}$$

which, in the original coordinates x,u, states that $X^*\Delta$ is a "combination" of Δ and of $X_Q u$ with its differential consequences. Now, another application of Lemma 2.1 (the role of y being played by Δ and $v_s^{(\ell)}$) gives that Δ must have the form

$$\Delta \;=\; R(s,z,v^{(m)})K(z,v^{(m)}) + \sum_\ell \Theta_\ell(s,z,v^{(m)})\,v_s^{(\ell)} \tag{22}$$

where the points to be emphasized are that R, K do not contain $v_s^{(\ell)}$ and that K does not depend *explicitly* on s.

If one now looks for solutions of $\Delta = 0$ which are independent on s, i.e. $v = w(z)$ and $v_s^{(\ell)} = 0$, then eq. (22) becomes a "reduced" equation $K^{(0)}(z, w^{(m)}) = 0$, just as in the exact symmetry case.

Remark 2. If X is a CS for a differential equation, then clearly also $X_\psi = \psi(x,u)X$, for any smooth function ψ, is another CS. While the invariant variables z are the same for X and X_ψ, the variable s turns out to be different. This implies that, writing the differential equation in terms of the canonical variables, one obtains in general different equations for different choices of ψ. All these equations will produce the same reduced equation when one looks for invariant solutions $v = w(z)$.

Example 3. It is known that the nonlinear acoustic equation [12, 13, 26]

$$u_{tt} \;=\; u\,u_{xx} \qquad ; \qquad u = u(x,t)$$

admits the CS

$$X \;=\; 2t\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + 8t\frac{\partial}{\partial u}$$

Introducing the canonical variables $s = t, z = x - t^2$ and $u = v(s,z) + 4s^2$, the equation becomes

$$8 - 2v_z - vv_{zz} + v_{ss} - 4sv_{sz} \;=\; 0 \tag{23}$$

Considering instead $(1/2t)X = \partial/\partial x + (1/2t)\partial/\partial t + 4\partial/\partial u$, one gets $s = x$, $z = x - t^2, u = v(s,z) + 4(s-z)$ and the equation becomes

$$8 - 2v_z - vv_{zz} - 4sv_{ss} + 4zv_{ss} - 2vv_{sz} - 8sv_{sz} + 8zv_{sz} - vv_{ss} \;=\; 0 \tag{24}$$

Both equations (23) and (24) have the form (22), as expected, and both become the reduced ODE $8 - 2w_z - ww_{zz} \;=\; 0$. $\odot$

The presence of some terms containing s in the above equations (23,24) shows that X is not an exact symmetry, and the fact that these terms disappear when $v_s = 0$ shows that X is a CS.

However, the above one is not the only way to obtain reduced equations.

Indeed, the rather disappointing remark is that, as pointed out by Olver and Rosenau [26] (see also [25]), given an *arbitrary* vector field X, if one can find some particular simultaneous solution $\widehat{u}$ of the two equations (17), then the CS condition (18) turns out to be automatically satisfied when evaluated along this solution, i.e.:

$$X^*(\Delta)|_{\widehat{u}} \,=\, 0 \ . \tag{25}$$

It can be interesting to verify this fact in terms of the canonical variables s, z, v: indeed one has ($\mathrm{d}/\mathrm{d}s$ is the total derivative)

$$X^*(\Delta) \,=\, \frac{\partial \Delta}{\partial s} \,=\, \frac{\mathrm{d}\Delta}{\mathrm{d}s} - v_s^{(\ell)} \frac{\partial \Delta}{\partial v^{(\ell)}} \tag{26}$$

which vanishes if one chooses a solution of $\Delta = 0$ of the form $\widehat{v} = \widehat{w}(z)$. Even more, it is enough to find an *arbitrary* solution of $\Delta = 0$; then, choosing any vector field leaving invariant this solution, one could conclude that – essentially – any vector field is a CS, and any solution is invariant under some CS: cf. [26] ! This issue has been also considered in [32], from another point of view (see also the end of this section).

The point is that the existence of some solution $\widehat{u}$ of the two equations (17) is not exactly equivalent to the condition (18), this happens essentially because X in this case is a symmetry of an *enlarged* system which includes the compatibility conditions of the differential consequences of both equations in (17) (or the "integrability conditions": see [2],[25]–[27]). Therefore it is important to clearly distinguish *different* notions and introduce a sort of "classification" of CS.

We will say that X is CS in "true" or standard sense if $X^*(\Delta)|_\Sigma \,=\, 0$ is satisfied: the discussion and the examples above cover precisely this case; also the examples of CS considered in the literature are usually CS of standard type (see e.g. [8]-[13], [28]-[31], but see also [19, 26, 32, 33]). Instead, when $X^*(\Delta)|_{\widehat{u}} \,=\, 0$ is satisfied only for some $\widehat{u}$, we shall say that a "weak" CS is concerned (we will be more precise in a moment; notice however that some authors call generically "weak" symmetries all non-exact symmetries). What happens in this case is – once again – more clearly seen in the canonical variables determined by the given vector field X (see also [15]): assume indeed for a moment that in these coordinates the PDE takes the form

$$\Delta \,=\, \sum_{r=1}^{\sigma} s^{r-1} K_r(z, v^{(m)}) + \sum_{\ell} \Theta_\ell(s, z, v^{(m)})\, v_s^{(\ell)} \,=\, 0 \tag{27}$$

where the part not containing $v_s^{(\ell)}$ is a polynomial in the variable s, with coefficients K_r not depending explicitly on s. Now, if one looks for $X-$invariant solutions $w(z)$ of $\Delta = 0$, one no longer obtains reduced equations involving only the invariant variables z and $w(z)$, as in the case of Eq. (22), but one is

faced (cf. [26, 27]) with the system of reduced equations (not containing s nor functions of s)

$$K_r^{(0)}(z, w^{(m)}) \;=\; 0 \qquad ; \qquad r = 1, \ldots, \sigma \tag{28}$$

Assume that this system admits some solution (it is known that the existence of invariant solutions is by no means guaranteed in general, neither for "true" CS, nor for "exact" Lie symmetries), and denote by Σ_σ the set of these solutions: for any $\widehat{w}(z) \in \Sigma_\sigma$ we are precisely in the case of weak CS.

The identical conclusion holds if the initial PDE is transformed into an expression of this completely general form (instead of (27))

$$\Delta \;=\; \sum_{r=1}^{\sigma} R_r(s, z, v^{(m)})\, K_r(z, v^{(m)}) + \sum_{\ell} \Theta_\ell(s, z, v^{(m)})\, v_s^{(\ell)} \;=\; 0 \tag{29}$$

with the presence here of a sum of σ terms $R_r K_r$ (with $\sigma > 1$), where the coefficients R_r which depend on s are grouped together, with the only obvious condition that the coefficients R_r be linearly independent (the idea should be that of obtaining the minimum number of independent conditions (28)).

We now see that the set Σ_σ can be characterized equivalently as the set of the solutions of the system

$$\Delta \;=\; 0 \quad ; \quad \frac{\partial \Delta}{\partial s} \;=\; 0 \quad ; \ldots ; \quad \frac{\partial^{\sigma-1}\Delta}{\partial s^{\sigma-1}} \;=\; 0; \quad v_s^{(\ell)} \;=\; 0 \tag{30}$$

(indeed the R_r are also functionally independent as functions of s).

Conversely, if a $\Delta(s, z, v^{(m)}) = 0$ is such that a system like (30) admits the symmetry $X = \partial/\partial s$, then condition (9) must be satisfied, and applying once again Lemma 2.1, we see that Δ must have the form (29).

Therefore (29) is the most general form of an equation exhibiting the *weak* CS $X = \partial/\partial s$, to be compared with (22), which corresponds to the case of *true* CS.

Let us now come back to the original coordinates x, u: we will see that the set of conditions (30) is the result of the following procedure.

Given the equation $\Delta = 0$, and a vector field X, assume that the system of equations (17) is *not* symmetric under X (therefore, that X is not a "true" CS for $\Delta = 0$), then put

$$\Delta^{(1)} := X^*(\Delta) \tag{31}$$

and consider $\Delta^{(1)} = 0$ as a new condition to be fulfilled, obtaining in this way the augmented system (the first step of this approach is similar to a procedure, involving contact vector fields, which has been proposed in [34])

$$\Delta \;=\; 0 \qquad ; \qquad \Delta^{(1)} \;=\; 0 \qquad ; \qquad X_Q u \;=\; 0 \tag{32}$$

If this system is symmetric under X, i.e. if

$$X^*(\Delta)|_{\Sigma_1} \;=\; 0 \tag{33}$$

where Σ_1 is the set of simultaneous solutions of (32), we can say that X is "weak CS of order 2" (according to this, a true CS is of order 1). If instead (32) is not symmetric under X, the procedure can be iterated, introducing

$$\Delta^{(2)} := X^*(\Delta^{(1)}) \tag{34}$$

and appending the new equation $\Delta^{(2)} = 0$, and so on. Finally, we will say that X is a weak CS of order σ if

$$X^*(\Delta)|_{\Sigma_\sigma} = 0 \tag{35}$$

where Σ_σ is the set (if not empty, of course) of the solutions of the system

$$\Delta = \Delta^{(1)} = \ldots = \Delta^{(\sigma-1)} = 0 \quad , \quad X_Q\, u = 0 \tag{36}$$

(as already pointed out, it is understood – here and in the following – that also the differential consequences of $X_Q u = 0$ must be taken into account; clearly, the additional conditions $\partial\Delta/\partial s = 0$ or $X^*\Delta = 0$ and so on, should not be confused with the differential consequences of the equation $\Delta = 0$).

Remark 3 (The "partial" symmetries). The above procedure for finding weak CS is reminiscent of the procedure used for constructing *partial symmetries*, according to the definition proposed in [14] (see also [35]), the (relevant !) difference being the presence in the weak CS case of the additional condition $X_Q u = 0$. Let us recall indeed that a vector field X is said to be a partial symmetry of order σ for $\Delta = 0$ if the set of equations, with the above definitions (31,34),

$$\Delta = \Delta^{(1)} = \ldots = \Delta^{(\sigma-1)} = 0 \tag{37}$$

admits some solutions. In terms of the variables s, z and $v(s, z)$, conditions (37) are the same as (30) but *without* the conditions $v_s^{(\ell)} = 0$. The set of solutions found in the presence of a partial symmetry provides a "symmetric set of solutions", meaning that the symmetry transforms a solution belonging to this set into a – generally different – solution in the same set. If, in particular, this set includes some solutions which are left fixed by X, then this symmetry is also a CS, either true or weak. So, we could call the weak CS, by analogy, "partial conditional symmetries of order σ".

We can now summarize our discussion in the following way.

Definition 3 *Given a PDE $\Delta = 0$, a projectable vector field X is a "true" conditional symmetry for the equation if it is a symmetry for the system*

$$\Delta = 0 \quad ; \quad X_Q u = 0 \; . \tag{38}$$

A vector field X is a "weak" CS (of order σ) if it is a symmetry of the system

$$\Delta = 0 \; ; \; \Delta^{(1)} := X^*(\Delta) = 0 \; ; \; \Delta^{(2)} := X^*(\Delta^{(1)}) = 0 \; ; \; \ldots ; \; \Delta^{(\sigma-1)} = 0 \; ;$$
$$X_Q u = 0 \; . \tag{39}$$

Proposition 1 *If X is a true CS, the system (38) gives rise to a reduced equation in $p-1$ independent variables, which – if admits solutions – produces $X-$invariant solutions of $\Delta = 0$. If X is a weak CS of order σ, the system (39) gives rise to a system of σ reduced equations, which – if admits solutions – produces $X-$invariant solutions of $\Delta = 0$. Introducing $X-$adapted variables s, z, such that $Xs = 1$, $Xz = 0$, the PDE has the form (22) in the case of true CS, or (29) in the case of weak CS.*

We can then rephrase the Olver-Rosenau statement [26] saying that *any vector field X is either an exact, or a true CS, or a weak CS*. Similarly, rewriting the equation $\Delta = 0$ as in (27) or in (29) but *without isolating* the terms $v_s^{(\ell)}$, we can also say, recalling the procedure used for obtaining partial symmetries, that *any X is either an exact or a partial symmetry*. It is clear however that, as already remarked, the set of solutions which can be obtained in this way may be empty, or contain only trivial solutions (e.g., $u = \text{const}$).

Example 4. It is known that Korteweg-de Vries equation

$$\Delta := u_t + u_{xxx} + uu_x \,=\, 0 \qquad\qquad u \,=\, u(x,t)$$

does not admit (true) CS (apart from its exact symmetries). There are however weak CS; e.g. it is simple to verify that the scaling vector field

$$X \,=\, 2x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$$

is indeed an exact symmetry for the system $\Delta = 0$, $\Delta^{(1)} = X^*(\Delta) = 0$ and $X_Q u = 0$, and therefore is a weak CS of order $\sigma = 2$, and $u = x/t$ is a scaling-invariant solution. But also, if we consider only the system $\Delta = 0$, $\Delta^{(1)} = X^*(\Delta) = 0$ (i.e. without the invariance condition $X_Q u = 0$), we obtain the symmetric set of solutions

$$u = \frac{x + c_1}{t + c_2} \qquad (c_1, c_2 \,=\, \text{const})$$

showing that X is also a partial symmetry. $\odot$

Few words, for completeness, about the so-called “direct method” [25, 32], [36]–[38] for finding solutions to PDE’s. The simplest and typical application of this method deals with PDE involving a function of two variables (which we shall call x, t, in view of the next applications), and one looks for solutions of the form (also called “similarity reduction solutions”)

$$u(x,t) \,=\, U(x,t,w(z)) \quad \text{with} \quad z = z(x,t) \tag{40}$$

or – more simply – of the form (according to a remark by Clarkson and Kruskal [36], this is not a restriction, see also Lou [37])

$$u(x,t) \,=\, \alpha(x,t) + \beta(x,t)\, w(z) \quad \text{with} \quad z = z(x,t) \tag{41}$$

one then substitutes (41) into the PDE and *imposes* that $w(z)$ satisfies an ODE. Although this method is not based on any symmetry, there is clearly a close and fully investigated relationship with symmetry properties; referring to [22, 24, 38] for a complete and detailed discussion, we only add here the following remark, to illustrate the idea in the present setting. Assuming in (41) that $z_t \neq 0$, one can choose x and z as independent variables, and then write (41) in the form $u = \widetilde{u}(x,z) = \widetilde{\alpha}(x,z) + \widetilde{\beta}(x,z)w(z)$. Then, putting

$$X \ = \ \xi(x,z)\frac{\partial}{\partial x} + \zeta(x,z)\frac{\partial}{\partial z} + \varphi(x,z,u)\frac{\partial}{\partial u} \tag{42}$$

one can fix $\zeta = 0$, in such a way that $X\,z = 0$, choose $\xi = 1$, and finally impose that $X_Q u \equiv \widetilde{\alpha}_x + \widetilde{\beta}_x w(z) - \varphi(x,z,u) = 0$ in order to determine the coefficient φ in (42). Then, by construction, $\widetilde{u}(x,z)$ is invariant under this X. It is also easy to see that the invariance condition $X_Q u = 0$ is satisfied exactly by the family (41). If $z_t = 0$ in (41), the same result is true retaining $z = x$ and t as independent variables, and choosing

$$X \ = \ \frac{\partial}{\partial t} + \varphi(x,t,u)\frac{\partial}{\partial u} \tag{43}$$

So, the direct method has produced a set of solutions to the given PDE which also satisfies the invariance condition $X_Q u = 0$; then, according to our discussion, X is a CS for the PDE: it is a true CS if $w(z)$ satisfies a single ODE, as is usually the case in the direct method, or a weak CS if the method has produced a separation of the PDE into a system of ODE's.

Notice that a generalization of this method has been proposed in [39], with the introduction of *two* functions of the similarity variable z; this procedure has been further extended in [32], where its relationship with method of differential constraints is also carefully examined.

Other reduction procedures, based on the introduction of suitable multiple differential constraints, have been also proposed, aimed at finding nonclassical symmetries and solutions of differential problems: see, e.g., [17],[40]-[42], and also [5]. It can be also remarked that in our discussion we have only considered the case of a single vector field X; clearly, the situation becomes richer and richer if more than one vector field is taken into consideration. First of all, the reduction procedure itself must be adapted and refined when the given equation admits an algebra of symmetries of dimension larger than 1 (possibly infinite): for a recent discussion see [43].

5 Examples from the Boussinesq equation

The symmetry properties of the Boussinesq equation

$$\Delta := u_{tt} + u_{xxxx} + u\,u_{xx} + u_x^2 \ = \ 0 \qquad ; \qquad u = u(x,t) \tag{44}$$

have been the object of several papers (see e.g. [10, 16, 17]), but it is useful to consider here some special cases to illustrate the above discussion. First of all,

let us give the *invariant* form (according to Theorem 1) of the equation under the (exact) dilational symmetry

$$D \;=\; x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} \tag{45}$$

with $s = \log x$, $z = x^2/t$ and $u = z\exp(-2s)\,v(s,z)$ we get

$$\begin{aligned}
&z^2(16z^2v_{zzzz} + 4zv_z^2 + 2v + 12v_{zz} + 4vzv_{zz} + z^2v_{zz} + 48zv_{zzz} + 2vv_z +\\
&4zv_z) - 6v_s - vzv_s + zv_s^2 + 4z^2v_sv_z + 11v_{ss} + zvv_{ss} + 4zv_{sz} + 4z^2vv_{sz} -\\
&6v_{sss} - 12zv_{ssz} + 24z^2v_{szz} + v_{ssss} + 8zv_{sssz} + 24z^2v_{sszz} + 32z^3v_{szzz} \;=\; 0
\end{aligned}$$

which indeed does not depend explicitly on s. The dilational-invariant solutions are found putting $v = w(z)$, and only the terms in parenthesis survive.

For what concerns "true" CS, writing the general vector field in the form

$$X \;=\; \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \varphi(x,t,u)\frac{\partial}{\partial u} \tag{46}$$

a complete list of CS has been given both for the case $\tau \neq 0$ (and therefore, without any restriction, $\tau = 1$) [10] and for the case $\tau = 0$ [16, 37], see also [44]; it has been also shown that the invariant solutions under these CS are precisely those found by means of the "direct method" [10, 16, 36, 37].

Let us give for completeness the form taken by the Boussinesq equation when rewritten in terms of the canonical variables determined by some of these CS. For instance, choosing the CS

$$X \;=\; t\frac{\partial}{\partial x} + \frac{\partial}{\partial t} - 2t\frac{\partial}{\partial u} \tag{47}$$

we get $s = t, z = x - t^2/2$ and $u = v(s,z) - s^2$, and the equation becomes

$$-2 - v_z + v_z^2 + vv_{zz} + v_{zzzz} + v_{ss} - 2sv_{sz} \;=\; 0 \tag{48}$$

Starting instead (cf. Remark 2) from $(1/t)X = \partial/\partial x + (1/t)\partial/\partial t - 2\partial/\partial u$, we get $s = x, z = x - t^2/2$, $u = v(s,z) + 2(z-s)$ and

$$\begin{aligned}
-2 - v_z + v_z^2 + vv_{zz} + v_{zzzz} + 2v_sv_z + vv_{ss} - 2sv_{ss} + 2zv_{ss} \quad &+ \\
2vv_{sz} - 4sv_{sz} + 4zv_{sz} + v_s^2 + v_{ssss} + 4v_{sssz} + 6v_{sszz} + 4v_{szzz} \quad &= \quad 0
\end{aligned} \tag{49}$$

Both equations (48) and (49) have the expected form (22), and become the known reduced ODE (cf. [10]) if one looks for solutions with $v = w(z)$ and $u = w(z) - t^2$.

In the case of CS with $\tau = 0$, the invariant solutions have the form (instead of (41))

$$u(x,t) \;=\; \alpha(x,t) + \beta(x,t)\,w(t) \tag{50}$$

where $w(t)$ depends only on t and satisfies an ODE. Choosing, e.g., (cf. [17])

$$X \;=\; \frac{\partial}{\partial x} + \Big(\frac{2u}{x} + \delta\frac{48}{x^3}\Big)\frac{\partial}{\partial u} \tag{51}$$

where δ may assume the values 0 or 1, the canonical variables are given by $s = x, z = t$, and $u(x,t) = -12\delta/x^2 + x^2 v(x,t)$, and the equation becomes

$$x^2(v_{tt}+6v^2)+8x^3vv_x+x^4v_x^2+12v_{xx}-12\delta v_{xx}+x^4vv_{xx}+8xv_{xxx}+x^2v_{xxxx}=0$$

which has the form (22), observing that the role of the variable s is played here by x; as expected, looking for solutions in which $v = w(t)$, this equation becomes one of the solutions listed in [37].

To complete the analysis, one can also look for solutions of the form

$$u(x,t) \;=\; \alpha(x,t) + \beta(x,t)\, w(x) \tag{52}$$

with $w(x)$ satisfying an ODE, or for CS of the form

$$X \;=\; \frac{\partial}{\partial t} + \varphi(x,t,u)\frac{\partial}{\partial u} \tag{53}$$

i.e. with $\xi = 0$. It is not difficult to verify that no true CS of this form is admitted by the Boussinesq equation. However, solutions of the form (52) can be obtained via *weak* CS. Indeeed, choosing e.g.

$$X \;=\; \frac{\partial}{\partial t} + \Big(\frac{1}{t^2} - \frac{2u}{t}\Big)\frac{\partial}{\partial u} \tag{54}$$

one now obtains $s = t, z = x$ and $u(x,t) = 1/t + v(x,t)/t^2$, giving

$$vv_{xx} + v_x^2 + 6v + t(v_{xx}+2) + t^2v_{xxxx} - 4tv_t + t^2v_{tt} \;=\; 0 \tag{55}$$

which is precisely of the form (29), showing that (54) is a weak CS (the role of s is played here by t). Looking indeed for solutions with $v = w(x)$, one gets (cf. (30)) a system of the three ODE's

$$vv_{xx} + v_x^2 + 6v = 0, \quad 2 + v_{xx} = 0, \quad v_{xxxx} = 0$$

admitting the common solution $w = -x^2$ and giving the solution $u = 1/t - x^2/t^2$ of the Boussinesq equation.

Another example of weak CS is the following

$$X \;=\; t^2\frac{\partial}{\partial x} + \frac{\partial}{\partial t} - \Big(2x + \frac{10}{3}t^3\Big)\frac{\partial}{\partial u} \tag{56}$$

now $s = t, z = x - t^3/3$ and $u = -2sz - s^4 + v(s,z)$. Instead of giving the form of the equation in these variables, let us now evaluate, according to our discussion (cf. (36)), the additional equations $\Delta^{(1)} = X^*(\Delta) = 0$ etc.: we get

$$\Delta^{(1)} \;=\; -10t - 3u_x - 2tu_{xt} - \frac{5}{3}t^3u_{xx} - xu_{xx} \;=\; 0 \tag{57}$$

$$\Delta^{(2)} \;=\; 2 + u_{xt} + t^2u_{xx} \;=\; 0 \tag{58}$$

The most general solution of the equation $\Delta^{(2)} = 0$ is

$$u \;=\; F(t) + G\Big(x - \frac{t^3}{3}\Big) - 2tx$$

where F, G are arbitrary functions of the indicated arguments; we easily conclude that (56) is a weak CS of order $\sigma = 3$ and, taking into account also the invariance condition $X_Q u = 0$, we obtain the invariant solutions

$$u(x,t) \;=\; -\frac{t^4}{3} - 2tx - \frac{12}{(x - t^3/3)^2} \quad \text{and} \quad u(x,t) \;=\; -\frac{t^4}{3} - 2tx + c$$

(c = const). If instead we do *not* impose the invariance condition $X_Q u = 0$ and solve – according to Remark 3 – the three equations (44,57,58), we find the slightly more general families of solutions

$$u(x,t) \;=\; -\frac{t^4}{3} - 2tx + c_1 t - \frac{12}{(x - t^3/3 - c_1)^2} \quad \text{and} \quad u(x,t) \;=\; -\frac{t^4}{3} - 2tx + c_2 t + c$$

which are transformed by the symmetry into one another, showing that (56) is also a partial symmetry for the Boussinesq equation.

6 Concluding remarks

Some of the facts presented in this paper were certainly already known, although largely dispersed in the literature, often in different forms and with different languages. This paper is an attempt to provide a unifying scheme where various notions and peculiarities of symmetries of differential equations can be stated in a natural and simple way. This allows us, in particular, to provide a precise characterization and a clear distinction between different notions of conditional symmetry: this is indeed one of the main objectives of our paper. We can also give a neat comparison between the notions of conditional, partial and exact symmetries; several new and explicit examples elucidate the discussion.

In the same unifying spirit, it can be also remarked that all the above notions can be viewed as particular cases of a unique comprehensive idea, which can be traced back to the general idea of appending suitable additional equations to the given differential problem $\Delta = 0$, *and* to search for (exact) symmetries of this augmented problem (cf. [45]). In other words, one looks for a supplementary equation, say $E = 0$, and a vector field X satisfying

$$X^*(\Delta) = G\Delta + HE \qquad ; \qquad X^*(E) = G_E\Delta + H_E E \tag{59}$$

(some authors call generically "conditional symmetries" for the equation $\Delta = 0$ all these symmetries, and call "Q-conditional" symmetries the more commonly named conditional symmetries.) Now, it is clear that all our above notions of symmetries simply correspond to different choices of the supplementary equation $E = 0$. Indeed:

i) if $E = 0$ is chosen to be $X_Q u = 0$, we are in the case of true CS,
ii) if $E = 0$ is given by the system $X^*(\Delta) = X^*(\Delta^{(1)}) = \ldots = 0$, we are in the case of partial symmetries,
iii) if $E = 0$ is the same as in ii) plus the condition $X_Q u = 0$, we are precisely in the case of weak CS.

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